ON THE THEORY OF TRANSMISSION BANDS OF PERIODIC WAVEGUIDES

(K TEORII POLOS PROPUSKANIIA PERIODICHESKIKH VOLNOVODOV)

PMM Vol.25, No.1, 1961, pp. 24-37

M.G. KREIN and G.Ia. LIUBARSKII
(Odessa, Khar'kov)

(Received July 16, 1960)

The propagation of acoustic waves of frequency ω in a waveguide can be described by the equation

$$\Delta\varphi + \frac{\omega^2}{c^2}\varphi = 0 \tag{0.1}$$

where ϕ is the velocity potential (v = grad ϕ), $c = c(x, y, z)$ is the sound velocity which depends on the properties of the medium at the point (x, y, z). The normal derivative vanishes on the boundary of the waveguide, namely

$$\frac{\partial\varphi}{\partial n} = 0 \tag{0.2}$$

In the present article periodic waveguides are considered. If l is the period of the waveguide, then a translation of the waveguide through a distance l along the x-axis will transform the waveguide into itself. This means that

$$c(x + l, \ y, \ z) = c(x, y, z)$$

Many problems involving electromagnetic waves in metallic guides can be reduced to an equation of the type (0.1). Let us consider, for example, a waveguide filled with some homogeneous dielectric and bounded by two metallic surfaces $y = y_1(x)$ and $y = y_2(x)$ $(-\infty < x < \infty)$. Denoting by ϕ the z-component of the electric field, we obtain Equation (0.1) where c is the velocity of light in the dielectric in the waveguide. In this case the following condition is satisfied on the boundary surfaces of the waveguide:

$$\varphi = 0 \tag{0.3}$$

The variable coefficient c can appear in Equation (0.1) also when one considers waveguide processes in a uniform medium. Let us consider a two-dimensional waveguide of variable cross-section in which the velocity of propagation is $c = 1$. The xy-plane of the waveguide can be considered as the plane of the complex variable z. Let us make a conformal representation $w = f(z)$, which transforms the region bounded by the walls of the waveguide into the strip $0 < \operatorname{Im} v < 1$ in such a way that the point at infinity remains fixed. The function $f(z)$ will be determined up to within an additive constant. It will, therefore, have the property that $f(z + l) = f(z) + \text{const}$, and the function $f'(z)$ will be a periodic function. In terms of the new variables u and $v (w = u + iv)$ Equation (0.1) will have the form

$$\frac{\partial^2\varphi}{\partial u^2} + \frac{\partial^2\varphi}{\partial v^2} + \frac{\omega^2}{|f'|^2}\varphi = 0$$

while the boundary condition (0.2), because the transformation is conformal, can be written in the form $\partial\phi/\partial v = 0$ when $v = 0$, and $v = 1$. Thus we again obtain an equation of the type (0.1) with a variable coefficient before ϕ.

The properties of waves in periodic waveguides are in many respects analogous to the solutions of ordinary linear differential equations with periodic coefficients. It seems that many concepts and methods used for the investigation of solutions of such equations can be applied with success to the study of periodic wave guides.

We note that for a two-dimensional waveguide bounded by two parallel lines $y = 0$ and $y = 1$ Equation (0.1) can be replaced by an approximating system of differential-difference equations with periodic coefficients:

$$\frac{d^2\varphi_n}{\partial x^2} - \sum_m A_{nm}\varphi_m + \lambda \sum_m P_{nm}(x)\varphi_m = 0 \qquad (n = 0, 1, \ldots, N) \tag{0.4}$$

where

$$\varphi_n(x) = \varphi\left(x, \frac{n}{N}\right), \qquad A_{nm} = -N^2(\delta_{n+1, m} - 2\delta_{nm} + \delta_{n-1, m})$$

$$P_{nm}(x) = \frac{\delta_{nm}}{c^2(x, n/N)}, \qquad \lambda = \omega^2$$

Depending on the type of the boundary conditions, we assume here that either $\phi_0 = \phi_N(x) = 0$ or $\phi_0(x) \equiv \phi_1(x)$, $\phi_N(x) \equiv \phi_{N-1}(x)$. We call attention to the fact that the matrices $\| A_{nm} \|$ and $\| P_{nm} \|$ are positive-definite.

A particular case of the system (0.4) with $A_{nm} \equiv 0$ is rather well investigated [1,2,3].

It is interesting to note that even though (0.4) is simpler in structure than Equation (0.1) some results obtained for Equation (0.1) cannot be carried over to the system (0.4).

In the sequel we shall consider only those solutions ϕ of Equation (0.1) which satisfy the boundary condition (0.2) or (0.3) and possess the following property:

$$\varphi(x+l,\ y,\ z)=e^{ikl}\varphi(x,\ y,\ z)$$

The last property is equivalent to the requirement that the function be expressible as the product of some periodic function $\phi(x,\ y,\ z)$ by e^{ikx}

$$\varphi(x,\ y,\ z)=e^{ikx}\psi(x,\ y,\ z),\qquad \psi(x+l,\ y,\ z)=\psi(x,\ y,\ z)\qquad (0.5)$$

Such solutions are natural generalizations of planar waves $\phi(x,y,z) = e^{ikx}\psi(y,\ z)$ which exist in uniform cylindrical waveguides. On the other hand, waves of the form (0.5) are analogous to those solutions of ordinary linear differential equations which occur in Floquet's theorem. The role of the multiplier is played by $\rho = e^{ikl}$. The number k is called the wave number.

We call attention to the fact that just as in the theory of differential equations so also in the theory of waveguides a special role is played by the multipliers which are equal to unity. Indeed, the solution of Floquet is bounded if the modulus of its multiplier is unity and it is unbounded in the opposite case. On the other hand, for waveguides, a solution of the type (0.5) has a physical sense if the multiplier $\rho = e^{ikl}$ is equal in modulus to unity, and it must be discarded (as not satisfying the boundary conditions with $x = \pm\infty$) if $|\rho| \neq 1$. Together with this similarity there exists also a great contrast. In the theory of waveguides it is important to determine whether a field of a given frequency ω can be propagated or, in other words, whether there exists at least one multiplier with modulus one for the given frequency ω. In the theory of differential equations with periodic coefficients it is important to find out whether all solutions of a given equation are bounded, i.e. whether all multipliers are unity in absolute value (in modulus).

Let us set ourselves the problem of finding those frequencies

$$\omega_1(k)\leqslant\omega_2(k)\leqslant\ldots\leqslant\omega_n(k)\leqslant\ldots,\qquad \operatorname{Im}k=0$$

for which Equation (0.1) has a solution of the type (0.5) satisfying the boundary condition (0.2) (problem A_1). The analogous problem with the boundary condition (0.3) we shall call problem A_2.

The function $\phi(x,\ y,\ z)$ determines the wave number k with an accuracy up to within the additive constant $2\pi/l$. Hence, the frequencies $\omega_n(k)$ are periodic functions of k with period $2\pi/l$. It is easy to see that $\omega_n(k)$ is an even function of k. The interval of values through which $\omega_n(k)$ runs when k varies from 0 to π/l is called the nth *transmission band*.

In the sequel we shall establish some properties of these transmission bands, especially of the first transmission band.

1. Maximal property of the frequencies $\omega_n(k)$. Let us consider a single "cell" V of a waveguide bounded by an arbitrary smooth surface S which intersects the waveguide, and a surface S' which is obtained from S by means of a parallel translation over a distance L along the x-axis. From the relation (0.5) it follows that at an arbitrary point $(\xi, \eta, \zeta,)$ lying on the surface S, and at the corresponding point $(\xi + l, \eta, \zeta)$ of the surface S' the following relations hold:

$$\varphi(\xi + l, \eta, \zeta) = e^{ikl}\varphi(\xi, \eta, \zeta), \quad \frac{\partial}{\partial n}\varphi(\xi + l, \eta, \zeta) = e^{ikl}\frac{\partial}{\partial n}\varphi(\xi, \eta, \zeta) \quad (1.1)$$

Conversely, every function which satisfies the conditions (1.1) can be continuously extended over the entire waveguide so that the function will satisfy condition (0.5).

Therefore, one can replace the problem $A_1(A_2)$ by the problem $A_1(S)$ $(A_2(S))$, which consists of finding the frequencies $\omega_n(k)$ and the functions ϕ_n satisfying Equation (0.1) inside the cell V, the boundary conditions (0.2) ((0.3)) on the lateral surfaces and the condition (1.1) on the surface S.

We note that under the indicated boundary conditions the boundary-value problems $A_1(S)$ and $A_2(S)$ are self-adjoint problems. If, additionally, we take into account the fact that

$$\int_V \varphi\triangle\varphi dv < 0, \qquad \varphi \not\equiv 0$$

(the function ϕ satisfies the boundary conditions (0.2) or (0.3) and the condition (1.1)) we can conclude that all the numbers $\omega_n(k)$ are real. We shall take them to be positive as is customary*.

For what follows it is convenient to introduce the notation

$$J_1\{u\} = \int_V |\mathrm{grad}\, u|^2 dv, \qquad J_2\{u\} = \int_V |u|^2 \frac{dv}{c^2(x, y, z)} \qquad (1.2)$$

* When $k = 0$, the problem A_2 has, obviously, the solution $\phi = \text{const}$, which corresponds to the frequency ω_1 which is equal to zero.

Since the numbers $\omega_n^2(k)$ are characteristic values of a self-adjoint boundary-value problem which by a known procedure can be reduced to a weighted integral equation, it follows that these numbers have the minimaximal properties. Namely

$$\omega_n^2(k) = \max_{(u_1, \ldots, u_{n-1})} \inf_{(u \perp u_1, \ldots, u_{n-1})} \frac{J_1\{u\}}{J_2\{u\}} \tag{1.3}$$

where the maximum is taken over arbitrary sets of $u_1, \ldots, u_{n-1}$ square-integrable functions, and the infimum is taken over all differentiable functions u which are orthogonal to the functions of the selected set

$$\int_V u\bar{u}_j \frac{dv}{c^2} = 0 \qquad (j = 1, \ldots, n-1) \tag{1.4}$$

and satisfy on the surface S the first of the boundary conditions (1.1). On the lateral surface of the cell V the function u is arbitrary in the case of problem $A_1(S)$. In the case of the problem $A_2(S)$ the function u satisfies the condition (0.3).

The maximum of the infimum is attained when $u_j = \phi_j$, $j = 1, \ldots, n-1$ and $u = \phi_n$.

Following Courant [4] it is possible to establish a number of properties of the spectrum by means of (1.3).

1. The characteristic frequencies $\omega_n(k)$ are monotone and depend continuously on the coefficient $\rho(x, y, z) = c^{-2}(x, y, z)$. Furthermore, the increment $\delta\omega_n^2(k)$ corresponding to the increment $\delta\rho$ satisfies the inequality

$$\left| \frac{\delta\omega_n^2(k)}{\omega_n^2(k)} \right| \leqslant \sup_{x, y, z} \left| \frac{\delta\rho(x, y, z)}{\rho(x, y, z)} \right|$$

If the increment $\delta\rho$ is of the same sign at all points, then the increment $\delta\omega_n^2(k)$ has the opposite sign.

2. Every deformation of the surface of a waveguide, which does not change the length of the period and which decreases the region V, leads to an increase of all the characteristic frequencies $\omega_n(k)$ of the problem $A_2(S)$ (and hence of the problem A_2).

3. The characteristic number $\omega_1(0)$ is a simple one, and the corresponding characteristic function is positive within the region V under the appropriate normalization.

The following asymptotic formula describes the gorwth of the numbers $\omega_n(k)$:

$$\lim_{n\to\infty}\frac{\omega_n(k)}{n^{1/3}}=\left[\frac{1}{6\pi^2}\int_V\frac{dv}{c^3}\right]^{-1/3}$$

2. Boundary of the first transmission band. Let us pose the question on the change of the frequency when one replaces the boundary conditions (1.1) on S by the conditions

$$\varphi(\xi,\ \eta,\ \zeta)=0,\qquad \varphi(\xi+l,\ \eta,\ \zeta)=0,\qquad (\xi,\ \eta,\ \zeta)\in S \tag{2.1}$$

or by the conditions

$$\frac{\partial\varphi(\xi,\ \eta,\ \zeta)}{\partial n}=0,\qquad \frac{\partial\varphi(\xi+l,\ \eta,\ \zeta)}{\partial n}=0,\qquad (\xi,\ \eta,\ \zeta)\in S \tag{2.2}$$

Let us denote the characteristic frequencies of the problems $A_i'(S)$ and $A_i''(S)$ ($i = 1,\ 2$) thus obtained by $\Omega_{in}(S)$ and $\omega_{in}(S)$. These frequencies also have the mini-maximal properties

$$\Omega_{in}^2(S)=\max_{(u_1,\ldots,u_{n-1})}\ \inf_{(v\perp u_1,\ldots,u_{n-1})}\frac{J_1\{v\}}{J_2\{v\}}$$

$$\omega_{in}{}^2(S)=\max_{(u_1,\ldots,u_{n-1})}\ \inf_{(w\perp u_1,\ldots,u_{n-1})}\frac{J_1\{w\}}{J_2\{w\}}$$

whereby on the surface S the function v satisfies the condition (2.1) while the function w is arbitrary; the functions $J_1(w)$ and $J_2(w)$ are determined in accordance with (1.2).

The class of functions v is smaller than the class of functions u, and this class in turn is smaller than the class of functions w. Therefore

$$\omega_n(S)\leqslant\omega_n(k)\leqslant\Omega_n(S) \tag{2.3}$$

i.e. the transmission band with the index n is contained among the characteristic numbers $\omega_n(S)$ and $\Omega_n(S)$ of the problems $A'(S)$ and $A''(S)$ (the index $i = 1,\ 2$ has been omitted).

Relation (2.3) was obtained by Vladimirskii [5] in 1946* and somewhat

---

* Vladimirski's derivation is based on the assumption that the set of characteristic frequencies of a very long resonator obtained from a waveguide with two partitions coincides practically with the transmission bands of the waveguide.

later by Karaseva and Liubarskii [6]*.

Naturally, there arises the question for what values of k the frequency $\omega_n(k)$ will take on the minimum or maximum values. One can also ask for what choice of the surface S will the frequency $\omega_n(S)$ attain its maximum value and for what choice of the surface S will the frequency $\Omega_n(S)$ take on its minimum value.

Let us consider these questions in relation to the first transmission band. Suppose that $\phi_1(x, y, z, \pi/l)$ is a wave which corresponds to the wave number $k = \pi/l$ ($\rho = e^{ikl} = -1$) and to the frequency $\omega = \omega_1(\pi/l)$. The function $\phi_1(x, y, z, \pi/l)$ is skewperiodic, i.e. $\phi_1(x + l, y, z, \pi/l) = -\phi_1(x, y, z, \pi/l)$. Furthermore, since the boundary conditions are real one can assume without restricting the generality that the function ϕ_1 is real. Therefore, there exists a surface σ which intersects the waveguide and at all of whose points the function $\phi_1(x, y, z, \pi/l)$ vanishes. If one chooses for S the surface σ, then the function $\phi_1(x, y, z, \pi/l)$ will be a solution of the problem $A'(S)$. Hence

$$\omega_1\left(\frac{\pi}{l}\right) = \Omega_1(\sigma)$$

Making use of this equation and relation (2.3) we obtain

$$\max_k \omega_1(k) = \omega_1\left(\frac{\pi}{l}\right), \qquad \min_S \Omega_1(S) = \Omega_1(\sigma) \tag{2.4}$$

This establishes the first part of the following theorem.

Theorem 2.1. The first transmission band is bounded from above by the frequency $\omega_1(\pi/l)$ which corresponds to the "π-wave" (i.e. to the skewperiodic function $\phi_1(x, y, z, \pi/l)$ and is bounded from below by the frequency $\omega_1(0)$ which corresponds to the function $\phi_1(x, y, z, 0)$.

---

* In the article of Karaseva and Liubarskii there is considered the propagation of electromagnetic waves of a general type (i.e. of waves which fail to have certain types of symmetry). The initial equation is rot rot $\mathbf{E} = (\omega^2/c^2)\mathbf{E}$, which for all $\omega \neq 0$ guarantees the fulfilment of the fourth equation of Maxwell, div $\mathbf{E} = 0$. If $\omega = 0$ the equation rot rot $\mathbf{E} = 0$ has infinitely many solutions, i.e. the frequency $\omega = 0$ is a degenerated value of infinite multiplicity, and relation (2.3) is not valid, which was pointed out by the authors of [6] after its publication. The difficulty that is connected with the indicated circumstance can be alleviated by narrowing the statement of the problem: namely, by starting with the initial equation $\Delta\mathbf{E} + (\omega^2/c^2)\mathbf{E} = 0$ and restricting the problems to those cases when this equation is reduced to a scalar form.

Let us prove the second part of the theorem. We consider the multiple period $L = Nl$, where N is an arbitrary integer. Let us denote by $\omega_n^*(k)$ those frequencies for which the differential equation (0.1) has a solution of the form

$$\Phi_n(x, y, z, k) = e^{ikx}\Psi_n(x, y, z, k) \qquad (2.5)$$
$$\Psi_n(x+L, y, z, k) = \Psi_n(x, y, z, k)$$

It is clear that all the functions

$$\varphi_n(x, y, z, k+2\pi m/L), \quad 0 \leqslant k \leqslant 2\pi/L, \qquad (m = 0, 1, \ldots, N-1) \qquad (2.6)$$

are contained among the function $\Phi_n(x, y, z, k)$. On the other hand, the functions (2.6) exhaust all possible functions (2.5), for in accordance with a theorem of Gel'fand [8] one can express any function whose square is integrable within the waveguide as a Fourier integral in terms of either the functions (2.5) or (2.6). In particular, the periodic function $\Phi_1(x, y, z, 0)$ which corresponds to the characteristic number $\omega_1^*(0)$ coincides with one of the functions

$$\varphi_1(x, y, z, 2\pi m/L) \qquad (m = 1, \ldots, N-1) \qquad (2.7)$$

Since one can assume without restricting the generality that the function $\Phi_1(x, y, z, 0)$ is positive, it is clear that it must coincide with the function $\phi_1(x, y, z, 0)$. All remaining functions (2.7) coincide with the functions $\Phi_n(x, y, z, 0)$. The frequencies which correspond to them are greater than the frequency $\omega_1(0)$

$$\omega_1(2\pi m/L) > \omega_1(0)$$

Since m/N is an arbitrary rational number smaller than unity, it follows from continuity that $\omega_1(0) < \omega_1(k)$, $k > 0$. This completes the proof of the theorem.

We note that we have shown simultaneously how one should select the surface S in order that the problem $A'(S)$ might have the smallest first characteristic number $\omega_1^2(S)$: one should take for S the nodal surface of the π-wave $\phi_1(x, y, z, \pi/l) = 0$.

Let us apply this rule to a cylinder of arbitrary cross-section by assuming the propagation velocity $c = c(y, z)$, namely, that c depends only on the coordinates y and z. In such a cylinder all waves of the type (0.5) are plane waves, i.e. they are of the following type:

$$\varphi(x, y, z, k) = e^{ikx}\psi(y, z)$$

Furthermore, in addition to the wave $\phi(x, y, z, k)$ there exists also the wave $\phi(-x, y, z, k)$.

It is easy to see that the function

$$\frac{1}{2}[\varphi(x, y, z, k)+\varphi(-x, y, z, k)]=\psi(y, z)\cos kx$$

is a π-wave when $k = \pi/l$. The nodal surfaces of this wave are planes x = const, which are perpendicular to the generator of the cylinder. Hence, the following theorem is true.

Theorem 2.2. Let C be a cylinder of volume V and having a cross-section x = const which is fixed in size and shape. The cylinder is bounded by two parallel surfaces S and S'. The first characteristic number $\omega_1(S)$ of the problem

$$\Delta\varphi+\frac{\omega^2}{c^2(y, z)}\varphi=0, \qquad \varphi=0 \quad \text{on } S \text{ and } S'$$

(on the lateral surface of the cylinder the function ϕ is zero or it has a vanishing normal derivative) will take on the smallest value if the surface S is a normal section of the cylinder.

If c = const this theorem follows directly from the principle of symmetrization presented in [7]. By means of the proper generalization of the principle one can obtain the present result.

3. Group velocity. An important physical characteristic of a waveguide is the group velocity*, namely, the derivative $d\omega/dk$. For the computation of the group velocity, when $k = k_0$, it is sufficient to know the function $\omega(k)$ in an arbitrarily small interval $(k_0, k_0 + \kappa)$ and then only with a accuracy up to the first-order terms in κ. Therefore, it is natural to use the methods of the theory of perturbation for the computation of the group velocity. The application of the theory of perturbations is simplified because of the mini-maximal properties of the frequencies $\omega_n(k)$.

We note, first of all, that the functions u, which may enter into relation (1.3), satisfy the first condition of (1.1). Therefore, the set M of admissible functions u is different for different k: $M = M(k)$. If u

---

* In the Appendix it will be shown that the functions $\omega_n(k)$, which are arranged in increasing order, are sectionally analytic functions; their analiticity is violated on the real k-axis only at those points where $\omega_n(k) = \omega_{n+1}(k)$ or $\omega_n(k) = \omega_{n-1}(k)$.

is an arbitrary function from the set $M(k)$, then the function $v = e^{i\kappa x}u$ obviously belongs to the set $M(k + \kappa)$. Conversely, every function v from $M(k + \kappa)$ can be represented in such a form. Hence, in view of (1.3), we have (in the notation of (1.2))

$$\omega_n^2 (k_0 + \varkappa) = \max_{(u_1, \ldots, u_{n-1})} \inf_{(u \perp u_1, \ldots, u_{n-1})} J_1 \{ue^{i\varkappa x}\} \qquad (J_2\{u\} = 1) \quad (3.1)$$

or

$$\omega_n^2 (k_0 + \varkappa) = \max_{(u_1, \ldots, u_{n-1})} \inf_{(u \perp u_1, \ldots, u_{n-1})} \{J_1 \{u\} +$$
$$+ i\varkappa \int_V \left(u \frac{\partial \bar{u}}{\partial x} - \bar{u} \frac{\partial u}{\partial x} \right) dv + \varkappa^2 \int_V |u|^2 dv \Big\} \qquad (J_2\{u\} = 1)$$

It is easy to show that the function $u = u_\kappa$ on which this maximum of the infimum is reached depends continuously on κ and hence differs little from the function $u = \phi_n(x, y, z, k_0)$. Therefore

$$2\omega_n (k_0) \frac{d\omega_n}{dk} = \frac{i}{J_2\{\varphi_n\}} \int_V \left[\varphi_n \frac{\partial \bar{\varphi}_n}{\partial x} - \bar{\varphi}_n \frac{\partial \varphi_n}{\partial x} \right] dv \qquad (3.2)$$

We note that under the integral sign in this equation there stands an expression which differs only by a normalization factor from the projection of the Umov-Poynting vector on the x-axis. The flux $S(\phi_n, k)$ of this vector is the same through every cross-section of the waveguide. Equation (3.2) can therefore be rewritten as

$$\frac{d\omega_n}{dk} = \frac{l}{2\omega_n(k_0)} S(\varphi_n, k), \qquad S(\varphi_n, k) = \frac{i}{J_2\{\varphi_n\}} \int_V \left[\varphi_n \frac{\partial \bar{\varphi}_n}{\partial x} - \bar{\varphi}_n \frac{\partial \varphi_n}{\partial x} \right] dy\, dz \qquad (3.3)$$

The group velocity is thus seen to be proportional to the flux of the Umov-Poynting vector, and hence to the energy flux.

We note that this fact is well known for the particular case when the waveguide is homogeneous and cylindrical.

We will call the multiplier $\rho_n = e^{ik}$ a multiplier of the first kind if $S(\phi_n, k) > 0$ and a multiplier of the second kind if $S(\phi_n, k) < 0$. Relation (3.3) shows that a multiplier of the first kind moves counterclockwise around the unit circle when the frequency ω_n is increasing; a multiplier of the second kind moves in the opposite direction under the same circumstances.

From the physical point of view a multiplier of the first kind corresponds to a wave which carries energy in the positive direction, while a multiplier of the second kind carries energy into the opposite direction.

Relations (3.2) and (3.3) require refinements when the point $k = k_0$ corresponds to several equal frequencies*,

$$\omega_n(k_0) = \omega_0 \qquad (n = m, m+1, \ldots, m+p-1)$$

i.e. in case of degeneration. Hereby there will correspond to the frequency ω_0 and to the wave number k_0 a space $\Phi_{m,p}(k_0)$ of functions $\phi(s, y, z, k_0)$. When $k = k_0$ the frequencies $\omega_n(k)$, $n = m, \ldots, m+p-1$ have no derivatives in general. Nevertheless, there exist left and right derivatives $\omega'_n(k_0 - 0)$ and $\omega'_n(k_0 + 0)$. They are given by

$$\omega_n'(k_0 \pm 0) = \frac{l}{2\omega_n(k_0)} S(\varphi_n^{\pm}, k_0) \tag{3.4}$$

where the $\phi_n \pm$ $(n = m, \ldots, m+p-1)$ are some functions from $\Phi_{m,p}(k_0)$. The function ϕ_n^+ has the following extremal property: if the function u belongs to $\Phi_{m,p}(k_0)$ and is orthogonal to the functions ϕ_m^+, ϕ_{m+1}^+, $\ldots$, ϕ_{n-1}^+, then the flux of the Umov-Poynting vector $S(u)$ has a minimum when $u = \phi_n^+$. Hence the problem of determining the function ϕ_n^+ in the space $\Phi_{m,p}(k_0)$ is an elementary algebraic problem. The function ϕ_n^- can be found in an analogous way. We note that all functions ϕ^- and ϕ^+ are connected by the relation $\phi_{m+r-1}^- = \phi_{m+p-r}^+$ $(r = 1, \ldots, p)$. Because to this the curve ω_{m+p-r} is, when $k > k_0$, a smooth continuation of the curve ω_{m+r-1} when $k < k_0$.

Relation (3.2) makes it possible to estimate the absolute value of the group velocity. Indeed

$$\left|\frac{d\omega_n}{dk}\right| \leqslant \frac{1}{\omega_n J_2\{\varphi_n\}} \int_V \left|\varphi_n \frac{\partial \varphi_n}{\partial x}\right| dv \leqslant \frac{1}{\omega_n J_2\{\varphi_n\}} \left(\int_V |\varphi_n|^2 dv \int_V |\operatorname{grad} \varphi_n|^2 dv\right)^{1/2} \leqslant$$
$$\leqslant \left(\frac{1}{J_2\{\varphi_n\}} \int_V |\varphi_n|^2 dv\right)^{1/2} \leqslant \max_{x, y, z} c(x, y, z)$$

i.e. the group velocity does not exceed the greatest local velocity of propagation of small signals

$$\left|\frac{d\omega_n}{dk}\right| \leqslant \max_{x, y, z} c(x, y, z) \tag{3.5}$$

From this follows the next estimate of the width of each transmission band:

---

* In this connection see the work of Vishik and Liusternik [10].

$$\Delta\omega_n \leqslant \frac{\pi}{l} \max_{x, y, z} c(x, y, z) \tag{3.6}$$

4. Collision of multipliers. Up to now the frequencies $\omega_n(k)$ ($n = 1, 2, \ldots$) have been considered as functions of a real variable k. However, another viewpoint is possible. Indeed, just as for real so also for complex values of ω one can define a set of multipliers $\rho_n = \rho_n(\omega)$ as those values for which Equation (0.1) with the corresponding boundary condition (0.2) or (0.3) has a nontrivial solution satisfying the condition $\phi(x + l, y, z) = \rho\phi(x, y, z)$.

Such an approach connects the theory of periodic waveguides with the existing theory of systems of differential equations with periodic coefficients, and thus facilitates the establishment of an analog between these theories.

It appears that there exist theorems which are analogs of known results in the theory of Poincaré and Liapunov [1,3].

Theorem 4.1. The multipliers $\rho_n(\omega)$ are distributed symmetrically relative to the unit circle if the quantity ω is real.

This proposition can be proved rigorously for multipliers $\rho_n(\omega)$ which are obtained by an inversion followed by an analytic continuation of the function $\omega_n(k)$ ($\rho = e^{ikl}$). For this it is sufficient to make use of the principle of symmetry in the theory of analytic continuation, on the basis of which there correspond to those values of ω which are symmetric to each other with respect to the real axis values of ρ which are symmetric relative to the unit circle.

Theorem 4.2. The multipliers $\rho_n(\omega)$ are distributed symmetrically relative to the real axis.

This follows from the fact that Equation (0.1) and also all boundary conditions are real. Hence, alongside the solution $\phi(x, y, z)$, there exists also a complex-conjugate solution $\bar{\phi}(x, y, z)$. These solutions correspond to complex-conjugate multipliers.

It follows from these symmetry properties that a multiplier can lie on the unit circle or the real axis only under the condition that there is another oppositely moving multiplier on the unit circle or on the real axis. Therefore, we have the next theorem.

Theorem 4.3. A multiplier cannot coincide with the unit circle or with the real axis unless it meets another multiplier.

Let us consider the meeting on the unit circle of two multipliers when

$\omega = \omega_0$. As ω increases these multipliers can either leave the circle or continue to move along the circle for some time. In the $k\omega$-plane there are certain graphs which correspond to these possibilities (see Fig. 1).

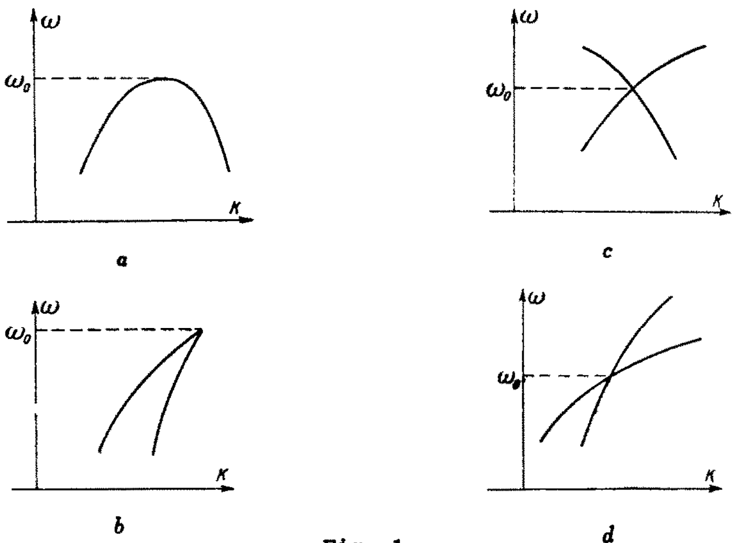

Fig. 1.

The cases (a) and (c) represent the meeting of two multipliers of different kinds, the cases (b) and (d) of two of the same kind. In cases (a) and (b) the multipliers leave the circle, while in cases (c) and (d) they remain on it. The case (b) can obviously not occur because the function $\omega_n(k)$ is definite and continuous for all real values of k. Hence the multiplier can leave the circle only in consequence of a collision with a multiplier of a different type.

In the cases of (c) and (d), with $\omega = \omega_0$, two transmission bands touch each other. We shall show that these cases are unstable in the sense that the slightest disturbance of the conditions of the problem will result in the disappearance of the point of intersection of the curves $\omega(k)$. Hereby one obtains in the plane (k, ω) curves which are represented in Fig. 2. We see that if two multipliers of different kinds meet then the slightest disturbance can cause the multipliers to leave the circle, and the frequency ω_0 will not belong to any transmission band. Relative to multipliers of the same kind, it can be said that the very fact of meeting (collision) is "unstable"; the slightest disturbance will cause the meeting not to occur.

An entirely analogous situation may also take place when several multipliers meet.

This situation does not differ in essence from the well-known quantum-mechanics effect of the non-intersection of two terms [9].

Let us establish instability in the cases (c) and (d), represented in Fig. 1, under a small change of the local velocity $b(x, y, z)$ which does not destroy the periodicity of the waveguide. Denoting the disturbance of the velocity by $c_0(x, y, z)$ we obtain

$$c_0(x, y, z) = c\,[1 + \varepsilon s(x, y, z)], \qquad \varepsilon \ll 1$$

$$|s(x, y, z)| < 1, \qquad |\operatorname{grad} s| < 1$$

Let us associate with each function u of $M(k)$ the function

$$u_0 = \frac{c_0}{c}\,u = u\,[1 + \varepsilon s(x, y, z)]$$

It is clear that the function u also belongs to the set $M(k)$ and that if u is normalized relative to the velocity c, then u_0 is normalized relative to the disturbed velocity c_0. If one substitutes u_0 for u the expression (3.1) takes on the following form:

$$[\omega_n^\circ\ (k_0 + \varkappa)]^2 = \max_{(u_1,\ldots,u_{n-1})} \inf_{(u \perp u_1,\ldots,u_{n-1})} \left\{ J_1\{u\} + 2\varepsilon \int_V s\,|\operatorname{grad} u|^2\,dv + \right. \tag{4.1}$$

$$\left. + 2\varepsilon \int_V (\bar{u}\operatorname{grad} u + u \operatorname{grad} \bar{u},\ \operatorname{grad} s)\ dv + i\varkappa \int_V \left[u\,\frac{\partial \bar{u}}{\partial x} - \bar{u}\,\frac{\partial u}{\partial x}\right] dv \right\} \qquad (J_2\{u\} = 1)$$

(here terms which are proportional to ϵ^2, $\epsilon\kappa$ and κ^2 have been dropped; J_1 and J_2 are given by (1.2)). We note that if $\epsilon = 0$, $\kappa = 0$, the maximum of the infimum is attained on an arbitrary function u_0 of $\Phi_{m,p}(k_0)$. One can prove that for small ϵ and κ the function u on which the maximum of the infimum (4.1) is attained differs but little from some function ϕ of $\Phi_{m,p}(k_0)$. Setting $u = \phi$ in (4.1) one can see that the integral $J_1(u)$ becomes equal to $\omega_n^2(k_0)$ and will not depend on the choice of $\phi \in \Phi_{m,p}(k_0)$;

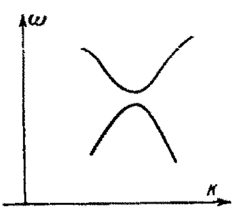

Fig. 2a.

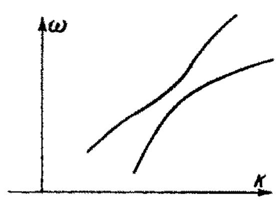

Fig. 2b.

the sum of the remaining three terms reduces to the sum of two Hermitian forms in the p-dimensional space

$$C(\varphi, \varphi) = \varepsilon A(\varphi, \varphi) + \varkappa B(\varphi, \varphi)$$

If the corresponding Hermitian matrix $C_{ik} = \epsilon A_{ik} = \kappa B_{ik}$ has multiple roots then the degeneration is preserved; in the opposite case we obtain the picture shown in Fig. 2.

Let us next make use of the following proposition of linear algebra.

There exist $m(m + 1)/2 - 1$ linear relations which are satisfied by the matrix elements of an arbitrary Hermitian matrix of the nth order that possesses at least one m-multiple characteristic number.

For the simplest (double) degeneration the number of these relations is two. Therefore, by changing only one parameter κ one cannot achieve these conditions for the matrix elements C_{ik} if the matrix B_{ik} is not of a special form.

5. Estimate of the boundary of the first transmission band. The minimal property of the frequencies $\omega_1(k)$ permits one to obtain the following lower estimate of the upper boundary $\omega_1(\pi/l)$ of the first transmission band under the assumption that the waveguide has the form of a cylinder of arbitrary cross section S. Namely

$$\omega_1^2\left(\frac{\pi}{l}\right) \geqslant \min_{y,z} \mu(y, z) \tag{5.1}$$

where $\mu(y, z)$ is the first characteristic number of the one-dimensional problem

$$\frac{d^2u}{dx^2} - \lambda_1 u + \mu\rho(x, y, z)\, u = 0, \qquad u(l) = -u(0)$$
$$u'(l) = -u'(0), \qquad \rho = c^{-2} \tag{5.2}$$

while λ_1 is the first characteristic number of the problem

$$\frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} + \lambda v = 0 \tag{5.3}$$

($v = 0$ on the boundary S in the case of problem A_2; in the case of problem A_1, the number λ_1 is zero).

For the proof of the inequality (5.1) we start from the obvious identity

$$\omega_1^2\left(\frac{\pi}{l}\right) = J_1\{\varphi_1\} = \int_V \left\{\left|\frac{\partial \varphi_1}{\partial x}\right|^2 + \lambda\,|\varphi_1|^2\right\} dv \qquad (J_2\{\varphi_1\} = 1) \tag{5.4}$$

where ϕ_1 is a characteristic function which corresponds to the frequency $\omega_1(\pi/l)$ and

$$\lambda = \int_V \left\{ \left| \frac{\partial \varphi_1}{\partial y} \right|^2 + \left| \frac{\partial \varphi_1}{\partial z} \right|^2 \right\} dv \Big/ \int_V |\varphi_1|^2 \, dv$$

From (5.4) it follows that

$$\omega_1^2 \left(\frac{\pi}{l} \right) \geqslant \inf_u \frac{1}{J_2\{u\}} \left(\int_V \left\{ \left| \frac{\partial u}{\partial x} \right|^2 + \lambda_1 |u|^2 \right\} dv \right) \geqslant$$

$$\geqslant \min_{(y,z)} \inf_u \left(\int_0^l \left\{ \left| \frac{\partial u}{\partial x} \right|^2 + \lambda_1 |u|^2 \right\} dx \Big/ \int_0^l |u|^2 \frac{dx}{c^2(x, y, z)} \right) \tag{5.5}$$

where

$$\lambda_1 = \inf_v \left(\int_S \left\{ \left| \frac{\partial v}{\partial y} \right|^2 + \left| \frac{\partial v}{\partial z} \right|^2 \right\} dy \, dz \Big/ \int_S |v|^2 \, dy dz \right) \tag{5.6}$$

The infimum of the right-hand side of Expression (5.5) is taken over all skewperiodic functions, and is nothing more than the first characteristic number μ_1 of the problem (5.2). The infimum (5.6) is taken over all functions v which satisfy on the boundary of the region S the condition $\partial v/\partial n = 0$ in the case of problem A_1, and the condition $v = 0$ in the case of problem A_2. In the first case $\lambda_1 = 0$, while in the second case λ_1 is the first characteristic number of the problem (5.3). The inequality (5.1) has thus been established.

Let us now consider the fact that the first characteristic number μ_1 of problem (5.2) coincides with the upper boundary of the central zone of stability. In the case of problem A_2, when $\lambda_1 = 0$, it follows from the classical criterion of Liapunov's stability that

$$\mu_1 \geqslant \frac{4}{l} \left[\int_0^l \frac{dx}{c^2(x, y, z)} \right]^{-1}$$

From this inequality follows the next lower estimate for the frequency $\omega_1(\pi/l)$:

$$\omega_1^2 \left(\frac{\pi}{l} \right) \geqslant \min_{y,z} \frac{4}{l} \left[\int_0^l \frac{dx}{c^2(x, y, z)} \right]^{-1} \tag{5.7}$$

which holds for the problem A_1.

Making use of other estimates of the upper bound of the central zone of stability one can obtain a large number of estimates for the frequency $\omega_1(\pi/l)$.

An estimate of the first transmission band in the case of problem A_1

(A_2) is given by the inequality

$$\omega_1(k) \leqslant \sqrt{\lambda(\rho^\circ, k)} \tag{5.8}$$

where $\lambda(\rho^\circ, k)$ is the first characteristic number of the problem

$$\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} + k^2 u + \lambda \rho^\circ u = 0; \qquad u = 0, \qquad \frac{\partial u}{\partial n} = 0 \text{ on the boundary } S$$

$$\rho^\circ(y, z) = \frac{1}{l} \int_0^l \frac{dx}{c^2(x, y, z)}$$

The inequality (5.8) follows from the obvious relation

$$\omega_1^2(k) \leqslant \inf_u \frac{J_1\{u(y, z)e^{ikx}\}}{J_2\{u(y, z)\}} = \inf_u \frac{\int\limits_S \{|\operatorname{grad} u(y, z)|^2 + k^2 |u|^2\} \, dy \, dz}{\int\limits_S |u(y, z)|^2 \left[\frac{1}{l} \int_0^l \frac{dx}{c^2(x, y, z)}\right] dy \, dz}. \tag{5.9}$$

By the same procedure one can obtain any convenient estimate $\omega_1(k)$ in the case of a circular cylinder of radius R. Choosing for the test functions u in (5.9) functions which depend only on the radius r of the cylinder, we obtain

$$\omega_1^2(k) \leqslant \inf_u \int_0^R \left\{\left|\frac{du}{dr}\right|^2 + k^2 |u|^2\right\} r dr \Big/ \int_0^R |u|^2 \rho(r) \, r \, dr \tag{5.10}$$

where

$$\rho(r) = \frac{1}{2\pi l} \int_0^l dx \int_0^{2\pi} \frac{d\varphi}{c^2[x, r, \varphi]} \tag{5.11}$$

Finally, replacing $\rho(r)$ by its smallest value ρ_m, we find that

$$\omega_1^2(k) \leqslant \frac{1}{\rho_m} \inf \int_0^R \left\{\left|\frac{du}{dr}\right|^2 + k^2 |u|^2\right\} r dr \Big/ \int_0^R |u|^2 r dr$$
$$= \frac{1}{\rho_m} \left[\left(\frac{2.405\ldots}{R}\right)^2 + k^2\right]$$

where 2.405... is the first root of the Bessel function $J_0(r)$. Thus, in the case of a circular cylinder, the lower boundary of the first transmission band $\omega_1(0)$ is bounded from above by the inequality

$$\omega_1(0) \leqslant \frac{2.405\ldots}{R} \left[\min_r \frac{1}{2\pi l} \int_0^l \int_0^{2\pi} \frac{d\varphi \, dx}{c^2[x, r, \varphi]}\right]^{-1/2} \tag{5.12}$$

6. Appendix. Analiticity of the function $\omega_n(k)$. The analiticity of the function $\omega_n(k)$ can be established with the aid of general theorems of the theory of perturbations [10,11] if the function ϕ vanishes on the surface of the waveguide, i.e. in the case of problem A_2. In the case of problem A_1 one can use the theory of perturbations only then when the surface of the waveguide is cylindrical. In this connection we shall first treat these two simple cases, and then later we shall indicate the proof of the analiticity of the functions $\omega_n(k)$ for all cases.

In the nature of the first step we make the substitution

$$\varphi(x, y, z, k_0 + \varkappa) = e^{-i\varkappa x}\psi(x, y, z) \tag{6.1}$$

It is easy to see that $\psi(x + l, y, z) = e^{ik_0 l}\psi(x, y, z)$. Hence the function $\psi(x, y, z)$ satisfies the boundary conditions (0.5) when $k = k_0$. It is also clear that the function ψ also satisfies the boundary condition (0.3) in the case of problem A_2. In the case of problem A_1 the function ϕ, generally speaking, does not satisfy the boundary condition (0.2). An exception is the case of a cylindrical waveguide in which the normal to the surface lies in the (y, z)-plane. Thus, in the two considered cases when $k = k_0 + \kappa$, the function $\psi(x, y, z)$ satisfies the same boundary conditions as the function $\phi(x, y, z, k_0)$. Furthermore, the following differential equation is satisfied:

$$\Delta\psi + 2ix\frac{\partial\psi}{\partial x} - \varkappa^2\psi + \frac{\omega^2}{c^2}\psi = 0$$

which is obtained through the substitution of (6.1) into (0.1).

Making one more substitution, $\psi/c = \chi$, we can rewrite the last equation in the form

$$A\chi + B\chi + \omega^2\chi = 0 \qquad \left(A\chi = c\Delta c\chi, \quad B\chi = 2i\varkappa c\frac{\partial c\chi}{\partial x} - \varkappa^2 c^2\chi\right)$$

It is easily seen that

$$\left[\int_V |B\chi|^2 dv\right]^{1/2} \leqslant K\left\{\left[\int_V |\chi|^2 dv\right]^{1/2} + \left[\int_V |A\chi|^2 dv\right]^{1/2}\right\}$$

where K is some constant. Therefore, by a theorem of Riesz-Nage [11, (Chapt. 9, Sect. 136)] the characteristic number ω_n^2 is an analytic function of κ and hence of k.

Let us consider next the case of a periodic waveguide. Let us indicate by L_2 the set of square-integrable functions inside the waveguide. The Laplace operator is defined on the dense, in L_2, set of differentiable functions which satisfy the boundary condition (0.2) (condition (0.3)); furthermore, the form $(\Delta\phi, \phi)$ is not positive ($\phi \in L_2$). Therefore, the operator $-\Delta + a (a > 0)$ has an inverse operator R_a. Applying the operator R_a to the δ-function $\delta(\mathbf{r} - \rho)$, we obtain Green's function

$$G_a(\mathbf{r}, \rho) = R_a \delta(\mathbf{r} - \rho)$$

The Green's function $G_a(\mathbf{r}, \rho)$ decreases exponentially as $|\mathbf{r} - \rho|$ increases. Hence, the series

$$G(\mathbf{r}, \rho, k) = \sum_{k=-\infty}^{\infty} G_a(\mathbf{r}, \rho + n\mathbf{l}) e^{ink} \qquad (\mathbf{r}, \rho \in V)$$

converges absolutely and uniformly in $\mathbf{r}$ and ρ and is analytic in k in some neighborhood of the real axis ($\mathbf{l}$ is here a vector of length l directed along the x-axis).

It is easily seen that $G(\mathbf{r}, \rho, k)$ is the Green's function of the operator $-\Delta + a$ operating on functions $u(x, y, z)$ defined within the cell V and satisfying the boundary conditions (0.2), (0.3) and (1.1).

The problem A_1 (A_2) is therefore equivalent to the integral equation

$$\varphi(r) = \int_V G(\mathbf{r}, \rho, k)\left[a + \frac{\omega^2}{c^2}\right] \varphi(\rho)\, d\rho$$

The analiticity of $\omega(k)$ now follows from the analiticity of the kernel $G(\mathbf{r}, \rho, k)$.

BIBLIOGRAPHY

1. Krein, M.G., Osnovnye polozheniia teorii λ-zon ustoichivosti kanonicheskoi sistemy lineinykh differentsial'nykh uravnenii s periodicheskimi koeffitsientami (Basic propositions of the theory of λ-zones of stability of canonical systems of linear differential equations with periodic coefficients). *Sb. pamiati A.A. Andronova (Collection in Memory of A.A. Andronov)*. Izd-vo Akad. Nauk SSSR, 1955.

2. Krein, M.G., O priznakakh ustoichivoi ogranichennosti reshenii periodicheskikh kanonicheskikh sistem (On criteria for the stable boundedness of solutions of periodic canonical systems). *PMM* Vol. 19, No. 6, 1955.

3. Gel'fand, I.M. and Lidskii, V.B., O strukture oblastei ustoichivosti lineinykh kanonicheskikh sistem differentsial'nykh uravnenii s periodicheskimi koeffitsientami (On the structure of the regions of stability of linear canonical systems of differential equations with periodic coefficients). *Uspekhi Matem. Nauk* Vol.10, No.1 (63), 1955.

4. Courant, R. and Hilbert, D., *Methods of Mathematical Physics*, Vol.I. (Russian translation). Moscow-Leningrad, 1951.

5. Vladimirskii, V.V., Rasprostranenie radiovoln vdol' tsepochki simmetrichnykh endovibratorov (Propagation of radio waves along a chain of symmetric endovibrators). *Dokl. Akad. Nauk* Vol. 52, No.3, 1946.

6. Karaseva, T.M. and Liubarskii, G.Ia., Polosy propuskaniia periodicheskikh volnovodov (Transmission bands of periodic waveguides). *Uch. zap. Khar'kovsk. Un-ta (trudy Fizich. otdeleniia)* Vol. 3, 1952.

7. Polya, G. and Szego, G., *Isoperimetric Inequalities in Mathematical Physics*. Princeton, 1951.

8. Gel'fand, I.M., Razlozhenie po sobstvennym funktsiiam uravneniia s periodicheskimi koeffitsientami (Expansions in terms of characteristic functions of an equation with periodic coefficients). *Dokl. Akad. Nauk* Vol. 73, No. 6, 1950.

9. Landau, L.D. and Lifshitz, E.M., *Kvantovaia mekhanika* (*Quantum Mechanics*). Moscow-Leningrad, 1948.

10. Vishik, M.I. and Liusternik, L.A., Vozmushchenie sobstvennykh znachenii znachenii i sobstvennykh elementov dlia nekotorykh nesamosopriazhennykh operatorov (Perturbation of characteristic values and characteristic elements for some non-selfadjoint operators). *Dokl. Akad. Nauk* Vol. 130, No. 2, 1960.

11. Riesz, F. and Nage, B., *Lectures on Functional Analysis*. (Russian translation). Moscow, 1954.

Translated by H.P.T.